Normalizable states through deformation of Lamé and the associated Lamé potentials

2008 J. Phys. A: Math. Theor. 41 285302

(http://iopscience.iop.org/1751-8121/41/28/285302)

IOP PUBLISHING — JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 285302 (11pp) doi:10.1088/1751-8113/41/28/285302

Normalizable states through deformation of Lamé and the associated Lamé potentials

S Sree Ranjani[1], **A K Kapoor**[1] **and P K Panigrahi**[2,3]

[1] School of Physics, University of Hyderabad, Hyderabad 500 046, India
[2] Indian Institute of Science Education and Research (IISER) Kolkata, Salt Lake, Kolkata 700 106, India
[3] Physical Research Laboratory, Navrangpura, Ahmedabad 380 009, India

E-mail: ssrsp@uohyd.ernet.in, akksp@uohyd.ernet.in and prasanta@prl.res.in

Received 28 December 2007, in final form 15 May 2008
Published 19 June 2008
Online at stacks.iop.org/JPhysA/41/285302

Abstract

We construct new aperiodic potentials with positive energy bound states starting from the Lamé and the associated Lamé potentials by confining them to the half line. The method used is the isospectral deformation discussed in supersymmetric quantum mechanics. The new potentials have normalizable eigenfunctions, with energies the same as certain levels of the original potentials on the half line. These states are similar to the von Neumann–Wigner states as they have an infinite number of zeros. The new potentials and their eigenfunctions are obtained numerically and their plots are given.

PACS number: 03.65Ge

1. Introduction

The existence of bound states in the continuum, where square integrable states with positive eigenvalues exist, have been studied by von Neumann and Wigner [1]. Unlike the conventional bound states these states have an infinite number of zeros. Here, we construct such states on the half line starting from the Lamé and the associated Lamé (AL) potentials. The method used is the isospectral deformation through supersymmetric quantum mechanics (SUSYQM) [2–4]. The constructed potentials are non-periodic, having eigenvalues the same as certain levels of the above potentials.

For our purpose, we start with the well-known family of periodic potentials on the full line namely, the Lamé and the associated Lamé potentials belonging to the class of elliptic potentials [5, 6]. These potentials are usually expressed in terms of the Jacobi elliptic functions $\mathrm{sn}(x, m)$, $\mathrm{cn}(x, m)$ and $\mathrm{dn}(x, m)$, where the parameter m is known as the elliptic modulus, with $0 < m < 1$ [7, 8].

On the real line, the Lamé potential

$$V(x) = j(j+1)m\ \mathrm{sn}^2(x,m), \tag{1}$$

is exactly solvable (ES) for integer values of j and for a given j there are $(2j+1)$ band-edges. The form of these band-edge solutions and explicit solutions for smaller values of j are given in [6, 9–11]. The Lamé potential has been proposed as a model for quasi 1D confinement of Bose–Einstein condensates (BEC) in a standing light wave [12].

The associated Lamé potential,

$$V(x) = a(a+1)m\ \mathrm{sn}^2(x,m) + b(b+1)m\frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2(x,m)}, \tag{2}$$

is exactly solvable when $a = b = j$ with j being an integer and quasi-exactly solvable (QES) when $a \neq b$. For small values of a and b the band-edge solutions have been obtained for both the ES and QES cases [6, 9–11, 13, 14]. For some QES examples only mid-band solutions are available [15].

In this paper, we construct new aperiodic potentials using the isospectral deformation technique discussed by Pappademos *et al*, in [16]. Here they construct quantum-mechanical bound states in a continuous energy spectrum using SUSYQM where the known eigenfunctions of the original potential are used. Application of the same technique to a periodic potential on the half line, deforms the potential making it non-periodic with square integrable solutions having same eigenvalues as the wavefunctions of the original potential. Earlier we used this technique to deform the singular Scarf potential whose eigenfunctions are in terms of the Jacobi polynomials [11, 17]. The deformed potential had one bound state and it had the same energy as that of the band-edge wavefunction of the original potential [18].

In the following section, we give an overview of SUSYQM and briefly describe the steps involved in isospectral deformation of a given potential and obtain the expressions for the deformed potential and its eigenfunctions. In sections 3 and 4, we construct the corresponding non-periodic potentials for the Lamé and the associated Lamé potentials, respectively. We give the plots of the new potentials and their eigenfunctions obtained numerically. In the last section, we present our conclusions.

2. Supersymmetric quantum mechanics

For a given 1D potential $V(x)$ with eigenfunctions $u_n(x)$ and eigenvalues $E_n (n = 0, 1, 2, \ldots)$, we can generate a new family of potentials $\tilde{V}(x;\lambda)$, isospectral to $V(x)$ [2–4]. The parameter λ is used to label the potentials in the isospectral family and takes values lying in the range $\lambda > 0$ and $\lambda < -1$. Setting $\hbar = 2m = 1$, we can write the superpotential as

$$W(x) = -\frac{u_0'(x)}{u_0(x)}. \tag{3}$$

The original potential $V(x)$ can be expressed in terms of $W(x)$ as

$$V(x) = W^2(x) - W'(x). \tag{4}$$

Its isospectral partner $V_+(x)$ is given by

$$V_+(x) = W^2(x) + W'(x). \tag{5}$$

Let $u_n^{(+)}(x)$ denote the eigenfunctions of $V_+(x)$, $(n = 1, 2, \ldots)$. We have $u_n^{(+)}(x) = Au_n(x)$, where $A = \mathrm{d}/\mathrm{d}x + W(x)$, (hence, $A^\dagger = -\mathrm{d}/\mathrm{d}x + W(x)$). Note that the ground state $u_0^+(x)$, cannot be obtained in this manner, as $Au_0(x) = 0$. Thus, $V_+(x)$ is isospectral to $V(x)$, except that its spectrum does not contain $u_0^{(+)}(x)$. Hence, to introduce the ground state into its spectrum and form a complete set of eigenstates, we need to find the most general superpotential $\tilde{W}(x)$, so that

$$V_+(x) = \tilde{W}^2(x) + \tilde{W}'(x). \tag{6}$$

Taking $\tilde{W}'(x) = W(x) + \phi(x)$ and substituting it into (6) gives the Bernoulli equation,

$$y' = 1 + 2W(x)y, \tag{7}$$

after the change of variable $y = 1/\phi(x)$. The solution of the above equation is

$$\frac{1}{y} = \phi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln(I_0 + \lambda), \tag{8}$$

where

$$I_0 = \int_{-\infty}^{x} u_0^2(y)\,\mathrm{d}y \tag{9}$$

and λ is a constant of integration. Thus, the most general superpotential is

$$\tilde{W}(x) = W(x) + \frac{\mathrm{d}}{\mathrm{d}x}\ln(I_0(x) + \lambda), \tag{10}$$

using which we obtain a family of potentials $\tilde{V}(x;\lambda)$ given by,

$$\begin{aligned}\tilde{V}(x;\lambda) &= \tilde{W}^2(x) - \tilde{W}'(x)\\ &= V(x) - 2[\ln(I_0 + \lambda)]''\\ &= V(x) - \frac{4u_0(x)u_0'(x)}{I_0 + \lambda} + \frac{2u_0^4(x)}{(I_0 + \lambda)^2}.\end{aligned} \tag{11}$$

The potential $\tilde{V}(x;\lambda)$ is isospectral to the potential $V(x)$ with the ground state [4]

$$\tilde{u}_0(x) = \frac{\sqrt{\lambda(1+\lambda)}u_0(x)}{(I_0 + \lambda)}. \tag{12}$$

Note that the above family of potentials also includes $V(x)$ because in the limit $\lambda \to \pm\infty$, $\tilde{V}(x;\lambda) \to V(x)$.

In standard SUSYQM, isospectral families of potentials which allow only bound states have been constructed, and $u_0(x)$ was the ground state. Generalization of this method, where we can use any non-singular eigenstate $u_n(x)$ of arbitrary energy E_n instead of $u_0(x)$, led to the construction of bound states in the continuum for *spherically symmetric potentials*. For more details the reader is referred to [16] and the references therein. We merely reproduce the statement of a theorem here.

Theorem 1. *Let $u_0(r)$ and $u_1(r)$ be any two non-singular solutions of the radial Schrödinger equation for the potential V(r) corresponding to arbitrarily selected energies E_0 and E_1 respectively, in the positive continuum region. Construct a new potential $\tilde{V}(r;\lambda)$ as prescribed by (11). Then, the two functions*

$$\tilde{u}_0(r) = \frac{u_0(r)}{I_0 + \lambda} \tag{13}$$

and

$$\tilde{u}_1(r) = (E_1 - E_0)u_1(r) + \tilde{u}_0(r)W_r(u_0(r), u_1(r)) \tag{14}$$

are solutions of the Schrödinger equation for the new potential $\tilde{V}(r;\lambda)$, corresponding to the same energies E_0 and E_1. Here $W_r(u_0(r), u_1(r))$ is the Wronskian. Due to the non-normalizability of the state $u_0(r)$, we have $\lambda > 0$ and $I_0 = \int_0^r u_0^2(y)\,\mathrm{d}y$.

Note that the original potential $V(r)$ had no integrable solutions but the new potential has one square-integrable solution $\tilde{u}_0(r)$ with energy E_0, along with non-normalizable eigenfunctions given by (14). The creation of the bound state can be elucidated by the fact that I_0 in (9) diverges owing to the non-integrability of u_0. Hence, as $I_0 \to \infty$, $\tilde{u}_0(r) \to 0$, resulting in a square-integrable wavefunction in the continuum.

A similar deformation of the potential $\tilde{V}(r;\lambda)$ with the non-normalizable state $\tilde{u}_1(r)$ gives another family of potentials $\tilde{\tilde{V}}(r;\lambda;\lambda_1)$, isospectral to both $V(r)$ and $\tilde{V}(r;\lambda)$, with two bound states $\tilde{\tilde{u}}_0(r)$ and $\tilde{\tilde{u}}_1(r)$ in the continuous spectrum with energies E_0 and E_1, respectively. The parameter λ_1 is a real number with $\lambda_1 > 0$.

The expressions for the new potential $\tilde{\tilde{V}}(r;\lambda;\lambda_1)$ and the two square-integrable states $\tilde{\tilde{u}}_0(r)$ and $\tilde{\tilde{u}}_1(r)$ are as follows:

$$\begin{aligned}\tilde{\tilde{V}}(r;\lambda;\lambda_1) &= \tilde{V}(r;\lambda) - 2[\ln(I_1+\lambda_1)]'' \\ &= \tilde{V}(r;\lambda) - \frac{4\tilde{u}_1(r)\tilde{u}_1'(r)}{I_1+\lambda_1} + \frac{2\tilde{u}_1^4(r)}{(I_1+\lambda_1)^2},\end{aligned} \tag{15}$$

$$\tilde{\tilde{u}}_0(r) = (E_0 - E_1)\tilde{u}_0(r) + \tilde{\tilde{u}}_1(r) W_r(\tilde{u}_1(r), \tilde{u}_0(r)) \tag{16}$$

and

$$\tilde{\tilde{u}}_1(r) = \frac{\tilde{u}_1(r)}{I_1+\lambda_1} \tag{17}$$

respectively, where

$$I_1 = \int_0^r \tilde{u}_1^2(y)\,\mathrm{d}y \tag{18}$$

and $W_r(\tilde{u}_1(r), \tilde{u}_0(r))$ is the Wronskian. Thus, we can create any number of bound states in the continuum by successive deformations of the potential $V(r)$ provided one knows the analytic expressions for the eigenfunctions and eigenvalues in the continuum.

In the following section, we apply the above technique to the Lamé and the associated Lamé potentials on the half line, with known eigenvalues and eigenfunctions. For this purpose, owing to the non-normalizability of the solutions of the periodic potential, we work on half line ($0 < x < \infty$). This allows us to exactly adopt the procedure described for spherically symmetric potentials, to one-dimensional periodic potentials and create new potentials. We see from the plots in the following sections that the deformed potential is aperiodic and it converges to the original potential for large x and in the limit $\lambda \to \infty$, the deformed potential tends towards the original potential. The difference between the two potentials is more pronounced near the origin. This deviation of the new potential from the original potential near the origin results in the occurrence of localized states. A similar situation resulted in the occurrence of bound states in the continuum [1]. Here the delicate interplay between the rate at which the oscillatory potential falls off and the time taken by the various maxima of the potential to interact was responsible for the creation of a bound state in the continuum [19, 20].

3. The Lamé potential

The Lamé potential in (1), with $j = 2$ and a constant added to make the ground state energy zero, is

$$V(x) = 6m\,\mathrm{sn}^2(x,m) - 2m - 2 + 2\delta, \tag{19}$$

where $\delta = \sqrt{1 - m + m^2}$. On the full line, this potential has two bands and a continuum. The expressions for the five band-edge wavefunctions and energies are [6, 9–11]

$$\psi_0(x) = 3m + 3 - \delta - 3m\ \mathrm{sn}^2(x, m), \quad E_0 = 2\delta - 2m - 2; \tag{20}$$

$$\psi_1(x) = \mathrm{cn}(x, m)\mathrm{dn}(x, m), \quad E_1 = m + 1; \tag{21}$$

$$\psi_2(x) = \mathrm{dn}(x, m)\mathrm{sn}(x, m), \quad E_2 = 4m + 1, \tag{22}$$

$$\psi_3(x) = \mathrm{cn}(x, m)\mathrm{sn}(x, m), \quad E_3 = m + 4, \tag{23}$$

$$\psi_4(x) = 3m + 3 - 3\delta - 3m\ \mathrm{sn}^2(x, m), \quad E_4 = 2\delta + 2m + 2, \tag{24}$$

with $\psi_0(x)$ and $\psi_1(x)$ representing the lower and upper band-edge wavefunctions of the first band and so on. As mentioned in the previous section, we examine only the half line problem. It needs to be pointed out that sticking to the half line problem changes the spectral characteristics of the above system. Only those states are allowed which vanish at the origin [21–23]. In the present case, only (22) and (23), which represent the lower and upper band-edge wavefunctions of the second band, satisfy the boundary condition, since $\mathrm{sn}(0, m) = 0$. Thus, on the half line, the Lamé potential with ($j = 2$) has only two known states namely, $\psi_2(x)$ and $\psi_3(x)$ with energies E_2 and E_3.

We follow the steps described in the previous section and construct a potential which has two bound states with energies E_2 and E_3. The entire procedure is done numerically and we give the plots of the deformed potential and its bound states.

First we deform the potential given in (19), using $\psi_2(x)$ in (22), to obtain the one-parameter potential with one bound state solution, which depends on the parameter λ. For this purpose, we first obtain I_0, whose plot is given in figure 1(*a*). As expected I_0 turns out to be a diverging integral. Using I_0 and equations (11), (13) and (14), we obtain and plot the deformed potential $\tilde{V}(x; \lambda)$ and the deformed wavefunctions $\tilde{\psi}_2(x)$ and $\tilde{\psi}_3(x)$ in figures 1(*b*)–(*d*), respectively. For comparison, the original potential and wavefunctions are plotted in dotted line.

It is clear from figure 1(*b*) that the deviation of the deformed potential from the original potential is more near the origin and it converges to the original potential for large x. From figures 1(*c*) and (*d*) we see that $\tilde{\psi}_2(x)$ is a normalizable state and $\tilde{\psi}_3(x)$ is not normalizable. Thus, with this deformation we have obtained a potential with only one bound state. In order to construct a potential with two bound states, we deform $\tilde{V}(x; \lambda)$ with $\tilde{\psi}_3(x)$ using (15)–(18). Plots of I_1, $\tilde{\tilde{V}}(x; \lambda; \lambda_1)$, $\tilde{\tilde{\psi}}_2(x)$ and $\tilde{\tilde{\psi}}_3(x)$ versus x are given in figures 2(*a*)–(*d*), respectively. We observe that compared to $\tilde{V}(x; \lambda)$, $\tilde{\tilde{V}}(x; \lambda; \lambda_1)$ deviates more from the original potential near the origin.

From figures 2(*c*) and (*d*), it is clear that both $\tilde{\tilde{\psi}}_2(x)$ and $\tilde{\tilde{\psi}}_3(x)$ are integrable. These states have energies E_2 and E_3, which are the energies of $\psi_2(x)$ and $\psi_3(x)$ respectively. In figures 3(*a*) and (*b*), we give the plots of the deformed wavefunctions $\tilde{\tilde{\psi}}_2(x)$ and $\tilde{\tilde{\psi}}_3(x)$ respectively, for two different values of λ and λ_1. (Without loss of generality we have set $\lambda = \lambda_1 = 1$ and $\lambda = \lambda_1 = 10$ in these figures.)

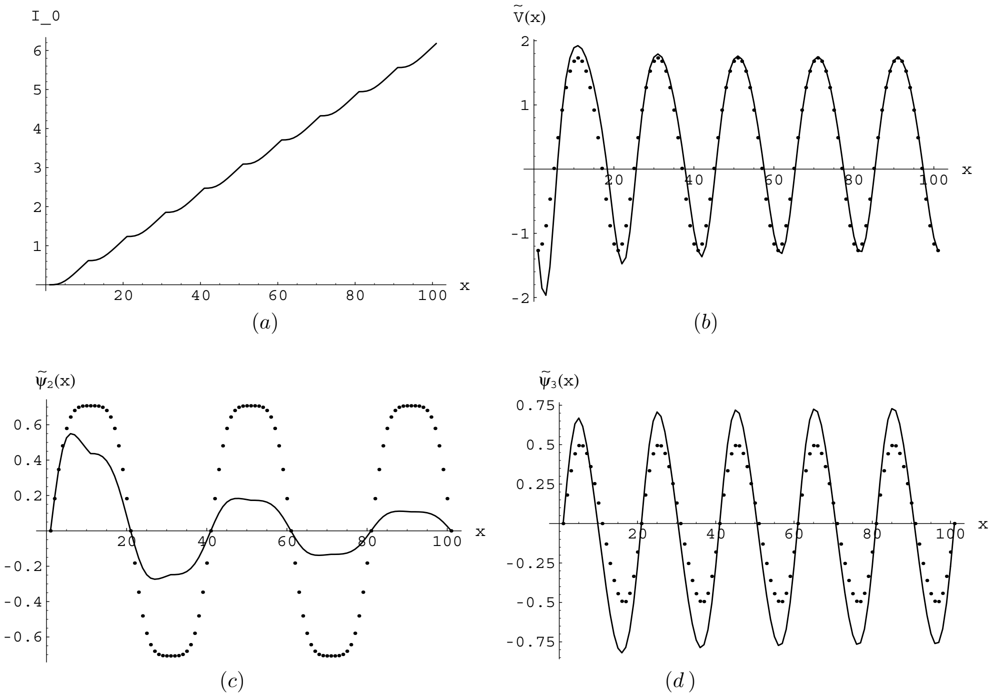

Figure 1. Lamé potential: first deformation using $\psi_2(x), \lambda = 1$. The dotted lines in each plot represent the plots of the corresponding original functions. (*a*) Diverging integral I_0, (*b*) the deformed potential $\tilde{V}(x;\lambda)$, (*c*) state $\tilde{\psi}_2(x)$, the normalizable state, (*d*) state $\tilde{\psi}_3(x)$, the non-normalizable state.

4. The associated Lamé potential

For the AL potential, we have considered the following two cases.

Case (i) $a = 7/2$ *and* $b = 1/2$. For these values of the potential parameters, the associated Lamé potential on the full line is QES with infinite number of bands. Of these infinite number of bands, analytical expressions for the band-edge eigenfunctions and eigenvalues of the lowest two bands and the continuum band-edge are known [6, 10, 11, 13, 15].

Case (ii) $a = 3/2$ *and* $b = 1/2$. In this case, the associated Lamé potential on the full line is QES and the analytical solutions for a pair of mid-band states alone are known [15].

For the above two cases we obtain non-periodic potentials after deforming them in the same way as done in the previous section. For both the cases, we provide only the plots of the deformed potential $\tilde{\tilde{V}}(x;\lambda;\lambda_1)$ and the two normalizable states, as the plots are very similar to the Lamé case.

Case (i). The expression for the potential is

$$V = \frac{63}{4} m\, \mathrm{sn}^2(x,m) + \frac{3}{4} m \frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2 x(x,m)} - 2 - \frac{29}{4} m + \delta_9 \tag{25}$$

where, $\delta_9 = \sqrt{4 - 4m + 25m^2}$. The known solutions are

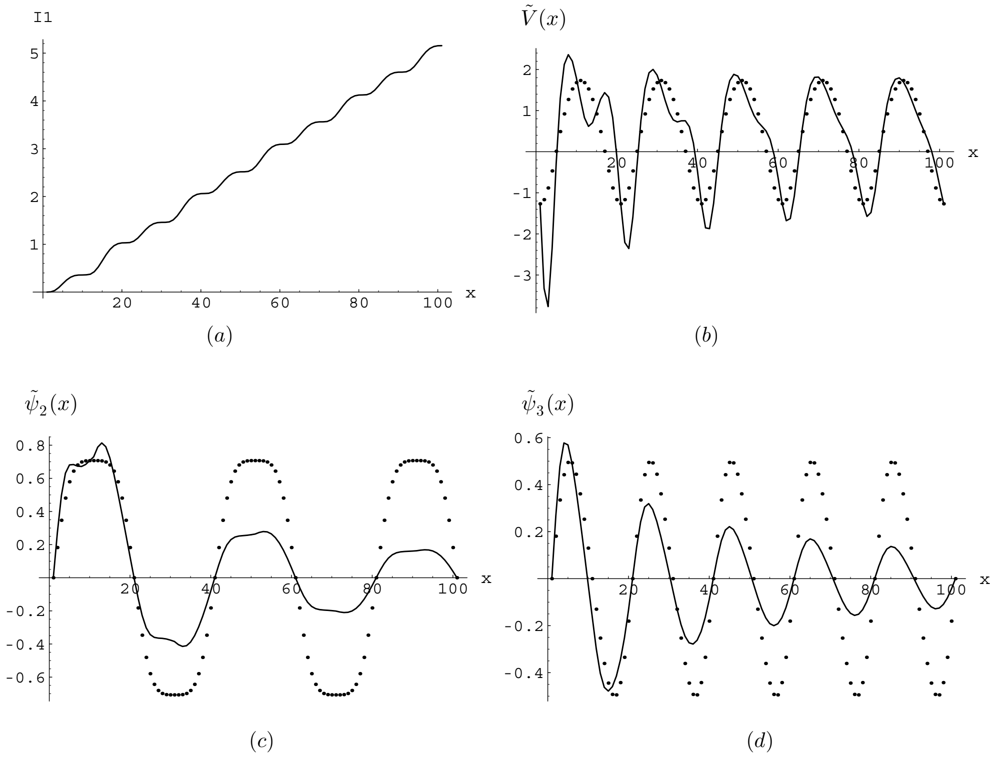

Figure 2. Lamé potential: second deformation using $\tilde{\psi}_3(x)$, $\lambda = \lambda_1 = 1$. The dotted lines in each plot represent the plots of the corresponding original functions. (*a*) Diverging integral I_1, (*b*) the doubly deformed potential $\tilde{\tilde{V}}(x;\lambda;\lambda_1)$, (*c*) state $\tilde{\tilde{\psi}}_2(x)$, normalizable state, (*d*) state $\tilde{\tilde{\psi}}_3(x)$, normalizable state.

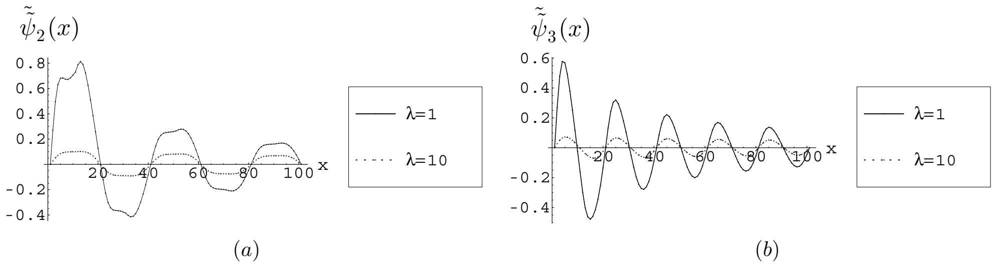

Figure 3. Comparison of the normalizable wavefunctions for different values of λ (bold line) and λ_1 (dotted line). For convenience $\lambda = \lambda_1$. The thick line is for $\lambda = \lambda_1 = 1$ and the dotted line is for $\lambda = \lambda_1 = 10$. (*a*) Deformed wavefunction $\tilde{\tilde{\psi}}_2(x)$, (*b*) deformed wavefunction $\tilde{\tilde{\psi}}_3(x)$.

$$\psi_0(x) = \mathrm{dn}^{3/2}(x,m)(12m\ \mathrm{sn}^2(x,m) - 5m - 2 - \delta_9),\quad E_0 = 0; \tag{26}$$

$$\psi_1(x) = \mathrm{cn}x(x,m)\,\mathrm{dn}^{3/2}(x,m)\mathrm{sn}(x,m),\quad E_1 = \delta_9 - m + 2; \tag{27}$$

$$\psi_2(x) = \mathrm{dn}^{3/2}(x,m)(12m\ \mathrm{sn}^2(x,m) - 5m - 2 + \delta_9),\quad E_2 = 2\delta_9; \tag{28}$$

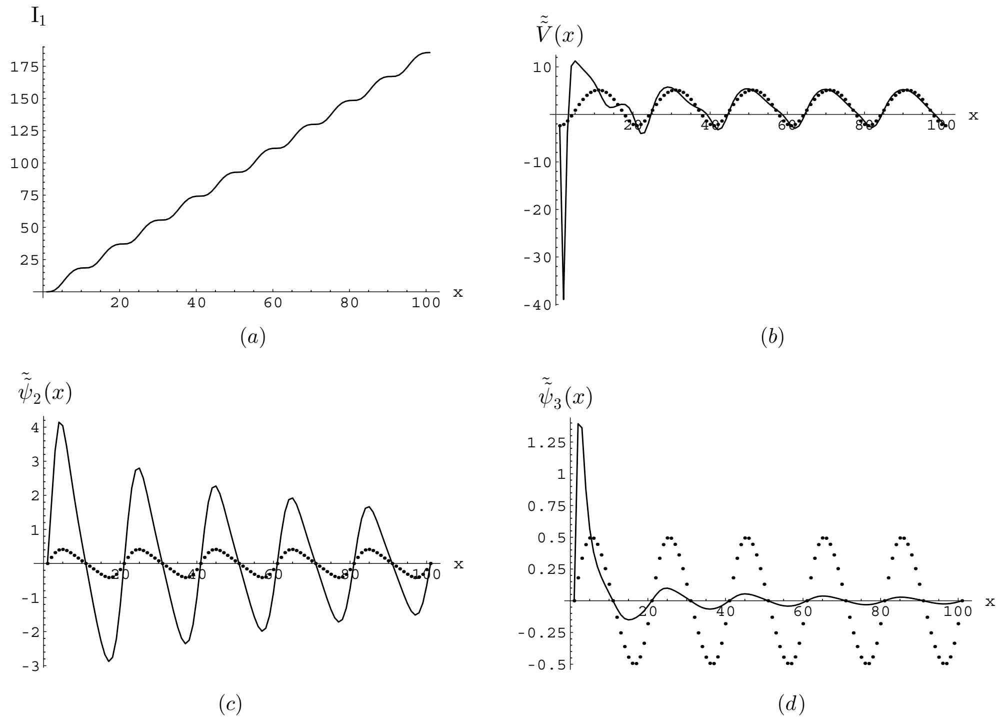

Figure 4. Associated Lamé potential case (i): second deformation using $\tilde{\psi}_3(x)$, with $\lambda = \lambda_1 = 1$. The dotted lines in each plot represent the plots of the corresponding original functions. (*a*) Diverging integral I_1, (*b*) the doubly deformed potential $\tilde{\tilde{V}}(x;\lambda;\lambda_1)$, (*c*) state $\tilde{\tilde{\psi}}_2(x)$, normalizable state, (*d*) state $\tilde{\tilde{\psi}}_3(x)$, normalizable state.

$$\psi_3(x) = \mathrm{cn}(x,m)\mathrm{dn}^{-1/2}(x,m)\mathrm{sn}(x,m)$$
$$(1 - 2\mathrm{sn}^2(x,m)),\ E_3 = 14 - 7m + \delta_9; \tag{29}$$

$$\psi_4(x) = 1 - 8\mathrm{sn}^2(x,m)\mathrm{cn}^2(x,m),$$
$$E_4 = 14 - 7m + \delta_9. \tag{30}$$

Note that the last two states, namely $\psi_3(x)$ and $\psi_1(x)$, are degenerate with the band-gap being zero. Considering the problem on half line implies that only the wavefunctions which satisfy the condition $\psi(0) = 0$ are valid. Thus, the states $\psi_1(x)$ and $\psi_3(x)$, which alone satisfy the boundary condition, are the solutions on the half line and hence are used to deform the above potential.

The first deformation is performed using the state $\psi_1(x)$ and the resulting potential $\tilde{V}(x;\lambda)$ has one bound state $\tilde{\psi}_1(x)$ with energy E_1 along with a non-normalizable state $\tilde{\psi}_3(x)$ with energy E_3.

Next the potential $\tilde{V}(x;\lambda)$ is deformed using the non-square integrable state $\tilde{\psi}_3(x)$ so that it can accommodate two bound states $\tilde{\tilde{\psi}}_1(x)$ and $\tilde{\tilde{\psi}}_3(x)$ which have energies E_1 and E_3, respectively. The corresponding plots for I_1, $\tilde{\tilde{V}}(x;\lambda;\lambda_1)$, $\tilde{\tilde{\psi}}_1(x)$ and $\tilde{\tilde{\psi}}_3(x)$ are given in figures 4(*a*)–(*d*).

In the above case, the potential was deformed using known solutions resulting in a new potential with bound states having energies corresponding to certain levels of the original

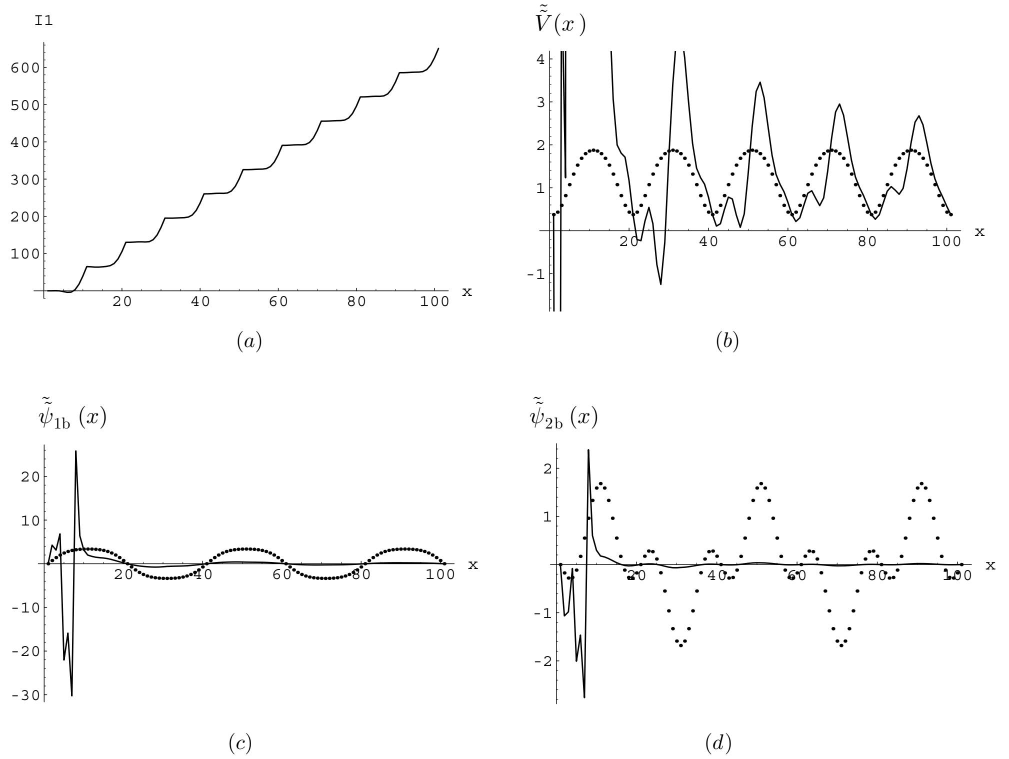

Figure 5. Associated Lamé potential case (ii): second deformation using $\tilde{\psi}_{2b}(x)$, with $\lambda = \lambda_1 = 1$. The dotted lines in each plot represent the plots of the corresponding original functions. (*a*) Diverging integral I_1, (*b*) the doubly deformed potential $\tilde{\tilde{V}}(x;\lambda;\lambda_1)$, (*c*) state $\tilde{\tilde{\psi}}_2(x)$, normalizable state, (*d*) state $\tilde{\tilde{\psi}}_3(x)$, normalizable state.

potential. In the next case, we see that the new deformed potential has an eigenstate which has the energy corresponding to the eigenstate, inside a band, of the original full line potential.

Case (ii). The associated Lamé potential with $a = 3/2$ and $b = 1/2$ is

$$V = \frac{15}{4}m\ \mathrm{sn}^2(x,m) + \frac{3}{4}m\frac{\mathrm{cn}^2(x,m)}{\mathrm{dn}^2 x(x,m)}. \tag{31}$$

For this potential only a pair of mid-band states given below are known analytically.

$$\begin{aligned}\psi_{1a}(x) &= \frac{\mathrm{cn}(x,m)(4-m-2\mathrm{sn}^2(x,m))}{\mathrm{dn}^{3/2}(x,m)},\\ \psi_{1b}(x) &= \frac{\mathrm{sn}(x,m)(4-2\mathrm{sn}^2(x,m))}{\mathrm{dn}^{3/2}(x,m)}\end{aligned} \tag{32}$$

with energy $E_1 = 1 + \frac{9m}{4}$.

$$\begin{aligned}\psi_{2a}(x) &= \frac{\mathrm{cn}(x,m)\mathrm{sn}^2(x,m)}{\mathrm{dn}^{3/2}(x,m)},\\ \psi_{2b}(x) &= \frac{\mathrm{sn}(x,m)(2\mathrm{sn}^2(x,m)-1)}{\mathrm{dn}^{3/2}(x,m)}\end{aligned} \tag{33}$$

with energy $E_2 = 9 + \frac{m}{4}$. From the above equations we can see that the mid-band states $\psi_{1b}(x)$ and $\psi_{2b}(x)$ vanish at the origin. Thus the potential V in (31) has only two known solutions on the half line. Hence, we make use of these wavefunctions to deform the potential in (31) numerically and construct a new potential with bound states having the energies E_2 and E_3. The plots of the two normalizable states along with the potential after the second deformation and the diverging integral I_1 are given in figure 5. Note that the potential in figure 5(*b*) is non-singular and well behaved near the origin. In order to show the overall behaviour of $\tilde{\tilde{V}}(x; \lambda, ; \lambda_1)$ we have refrained from showing the peak values (the maximum is approximately at 800 in this case).

5. Conclusions

In this paper, we have shown that we can construct new non-periodic potentials starting from the Lamé and the associated Lamé potentials on the half line by deforming them using SUSYQM. The deformed potential has eigenfunctions with energies same as certain eigenvalues of the original potential and in the limit $\lambda \to \infty$ tends to the original potential. We point out here that the bound states obtained after deformation are not bound states in the usual sense as they have infinite number of zeros and are similar to the von Neumann–Wigner bound states in continuum.

The Lamé potential, as mentioned in the introduction, has been proposed as a model for quasi 1D confinement of Bose–Einstein condensates (BEC) in a standing light wave [12]. As possible applications of BECs for quantum computation are currently being explored, a deformed Lamé potential which is non-periodic may be of use. Moreover, this work may have physical relevance in the context of optical lattice induced periodic potentials in the radial Gross–Pitaevskii equation, when nonlinearity is small [24].

Acknowledgments

SSR thanks S Lakshmi Bala for useful comments and acknowledges support from the Department of Science and Technology, India, under project no. SP/S2/K-14/2000. SSR also acknowledges financial support provided by Council of Scientific and Industrial Research (CSIR), Government of India.

References

[1] von Neumann J and Wigner E 1929 *Phys. Z.* **30** 465–7
[2] Witten E 1981 *Nucl. Phys.* B **188** 513–54
[3] Dutt R, Khare A and Sukhatme U 1988 *Am. J. Phys.* **56** 163–8
[4] Khare A and Sukhatme U 1989 *J. Phys. A: Math. Gen.* **22** 2847–60
[5] Magnus W and Wrinkler S 1966 *Hills Equation* (New York: Interscience)
[6] Arscot F 1964 *Periodic Differential Equations* (Oxford: Pergamon)
[7] Hancock H 1958 *Theory of Elliptic Functions* (New York: Dover)
[8] Whittaker E and Watson G 1963 *A Course of Modern Analysis* (Cambridge: Cambridge University Press)
[9] Sree Ranjani S, Kapoor A K and Panigrahi P K 2004 *Mod. Phys. Lett.* A **19** 2047–58
[10] Khare A and Sukhatme U 1999 *J. Math. Phys.* **40** 5473–94
[11] Sree Ranjani S 2005 *Thesis* The University of Hyderabad, India (submitted)
[12] Bronski J, Carr L, Deconinck B and Kutz J 2001 *Phys. Rev. Lett.* **86** 1402–05
[13] Sree Ranjani S, Kapoor A K and Panigrahi P K 2005 *Int. J. Theor. Phys.* **44** 1167–76
[14] Fernández D J and Ganguly A 2005 *Phys. Lett.* A **338** 203–8
[15] Khare A and Sukhatme U 2001 *J. Phys. A: Math. Gen.* **42** 5652–64
[16] Pappademos J, Sukhatme U and Pagnamenta A 1993 *Phys. Rev.* A **48** 3525–31 (*Preprint* hep-ph/9305336)

[17] Scarf F L 1958 *Phys. Rev.* **112** 1137–40
[18] Sree Ranjani S, Kapoor A K and Panigrahi P K 2006 *AIP Conf. Proc.* **864** 236–41
[19] Stillinger F H and Herrick D R 1975 *Phys. Rev.* A **11** 446–54
[20] Khelashvili A and Kiknadze N 1996 *J. Phys. A: Math. Gen.* **29** 3209–12
[21] Panigrahi P and Sukhatme U 1993 *Phys. Lett.* A **178** 251–7
[22] Marquez I, Negro J and Nieto L 1998 *J. Phys. A: Math. Gen.* **31** 4115–25
[23] Joannopoulos J, Meade R and Winn J 1995 *Photonic Crystals* (Princeton, NJ: Princeton University Press) p 50
[24] Baizakov B B, Malomed B A and Salerno M 2006 *Phys. Rev.* E **74** 066615–32